GENERAL AND COMPLETE SOLUTIONS OF THE EQUATIONS OF ELASTICITY

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In this paper we consider the problem of expressing a set of displacement components satisfying the homogeneous equations of elasticity in terms of harmonic functions [1-9].

The following definition of a "general" solution of the homogeneous equations of elasticity appears in [6]:

Definition 1. A solution of the equations of elasticity in terms of harmonic functions is said to be a *general* solution in a region D if for every set of displacement components satisfying the homogeneous equations of elasticity, for every closed region D' contained in the interior of D and for every $\epsilon > 0$ there exist functions $\phi_1{}^*$, $\phi_2{}^*$, $\phi_3{}^*$ (appearing in the given solution) harmonic in D such that the displacement components determined by $\phi_1{}^*$, $\phi_2{}^*$, $\phi_3{}^*$ satisfy the inequalities

$$|u - u^*| < \varepsilon, \qquad |v - v^*| < \varepsilon, \qquad |w - w^*| < \varepsilon$$

in D.

In other words, a given solution of the homogeneous equations of elasticity is a general solution in a region D, if it contains a sequence of functions $\phi_{1n}{}^*$, $\phi_{2n}{}^*$, $\phi_{3n}{}^*$ $(n = 1, 2, \ldots)$ harmonic in D such that the sequence of corresponding displacement components $u_n{}^*$, $v_n{}^*$, $w_n{}^*$ converges uniformly to the displacements u, v, w in D. This definition is of interest because such general solutions and well-known expansions of harmonic functions can be applied to special problems of the theory of elasticity.

Such general solutions can also be used to find approximate solutions for problems of elasticity (e.g. the variational method of Trefftz) and to determine the error of the approximate solutions [10-14].

It may of course happen that, for instance, the sequence $\phi_{1n}{}^*$ converges (as $n \to \infty$) to a function ϕ_1 which is not harmonic everywhere in D and that $\phi_1 = \lim \phi_{1n}^*$ may have singularities on certain curves in D. However, if the solution is a general one, the displacement components u_n^*, v_n^*, w_n^* and their derivatives converge uniformly to the displacements u, v, w and their derivatives.

It is appropriate at this point to introduce in addition the definition of a "complete" solution of the homogeneous equations of elasticity*.

Definition 2. A solution of the homogeneous equations of elasticity is called a *complete* solution in a region D, if every set of displacement components satisfying these equations in D can be expressed in terms of functions ϕ_1, ϕ_2, ϕ_3, ... harmonic everywhere in D which appear in the given solution. In other words, a given solution of the homogeneous equations of elasticity is a complete solution in D, if the harmonic functions ϕ_{1n}^*, ϕ_{2n}^*, ϕ_{3n}^* mentioned in the paragraph following Definition 1 converge uniformly to functions ϕ_1, ϕ_2, ϕ_3 harmonic everywhere in D. It is clear that a complete solution in a region D is also a general solution in the same region.

However, the converse is not true: a solution may be a general solution in D without being a complete solution. The concept of *general solution* is applicable to a wider class of regions than is the concept of *complete solution*.

We note that the author has proved in [6] that, with certain restrictions on the choice of the origin of coordinates and in some cases on Poisson's ratio, a whole series of solutions of the equations of elasticity, containing three harmonic functions, are general solutions for simply and doubly connected regions and has derived a general solution for multiply connected regions. These results are based on Definition 1 and an important theorem which states that a function harmonic in a simply connected region D_1 can be expanded in a uniformly convergent series of harmonic polynomials [15-19], if certain general restrictions are imposed on the surface S_1 which bounds D_1.

The above expansion theorem was proved for harmonic functions of two variables in the well-known papers of Runge and Walsh; and for harmonic functions of three variables, with varying restrictions of a general nature on the bounding surface S_1 of the region D_1 and on the harmonic

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* In [6] the term "general solution" is also used instead of "complete solution". It is clear from the context and the proofs of theorems, however, which meaning is to be ascribed to the term "general solution" in [6].

function on S_1, in the papers of Bergman [15], Szego [16], Keldysh and Lavrent'ev [17] and Vekua [18]. Vekua [18] proved the theorem on the assumption that S_1 is a Liapunov surface. If a harmonic function is continuous in the closed region $D_1 + S_1$, it can be approximated uniformly by harmonic polynomials in $D_1 + S_1$.

In this paper we shall show for which regions some well-known solutions (Papkovich [2], Grodskii [3], Neuber [5], etc.) containing three harmonic functions are incomplete (they are, however, general solutions in some of these regions) and for which regions these solutions are not general solutions (and are therefore not complete).

Our results generalize certain theorems proved in [1-7].

1. Neuber's solution. Neuber's solution [5] which contains three harmonic functions, can be obtained from Papkovich-Neuber's solution [2-5]

$$\begin{aligned} \mathbf{u} &= \mathbf{B} - \nu^{-1}\,\mathrm{grad}\,(\mathbf{r}\cdot\mathbf{B} + \varphi_0), \qquad \nu = 4(1-\sigma) \\ \mathbf{B} &= \mathbf{i}\varphi_1 + \mathbf{j}\varphi_2 + \mathbf{k}\varphi_3, \qquad \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \end{aligned} \tag{1.1}$$

(where ϕ_1, ϕ_2, ϕ_3 are functions, harmonic in a region D; x, y, z are the coordinates of points in D; $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors and σ is Poisson's ratio) by eliminating the function ϕ_3; that is, Neuber's solution is

$$\begin{aligned} u &= \varphi_1 - \nu^{-1}\,\partial F/\partial x, \qquad v = \varphi_2 - \nu^{-1}\,\partial F/\partial y \\ F &= x\varphi_1 + y\varphi_2 + \varphi_0 \qquad w = -\nu^{-1}\,\partial F/\partial z \end{aligned} \tag{1.2}$$

The following theorem is proved in [6] :

Theorem 1. In order that a set of displacement components u, v, w satisfying the homogeneous equations of elasticity in a region D be representable in the form (1.2), where ϕ_1, ϕ_2, ϕ_0 are functions harmonic everywhere in D, it is necessary and sufficient that for every harmonic function ϕ_3 there exist a function ψ_3 harmonic in D such that

$$\frac{\partial\psi_3}{\partial z} = \varphi_3, \qquad \nabla^2\psi_3 = \nabla^2\varphi_3 = 0 \tag{1.3}$$

This theorem clearly gives a necessary and sufficient condition that Neuber's solution be complete. For instance, let Ω^∞ be the unbounded region exterior to a sphere (see [6], 5) and let D^∞ be the region exterior to a closed surface S. Then if

$$\varphi_3 = \sum_{m=1,2,\ldots} \left(c_m + c_m'\,\frac{\partial}{\partial y}\right)\frac{\partial^m}{\partial x^m}\left(\frac{1}{r}\right) \tag{1.4}$$

where r is the distance from the center of the sphere to points of the region and c_m, c_m' are constants, there is no harmonic function ψ_3 satisfying (1.3) in Ω^∞ or D^∞. It is easily shown in the same way that there is no harmonic function ψ_3 satisfying (1.3) in the doubly connected region Ω_2 included between two concentric spheres if ϕ_3 is again given by (1.4).

Moreover, a much stronger theorem (important from the point of view of general solutions) holds for the regions Ω^∞ and Ω_2.

Theorem 2. It is possible to construct a function ϕ_3 harmonic in Ω^∞ or Ω_2 such that the inequality

$$|\partial\psi_3 / \partial z - \varphi_3| > \varepsilon_0 \tag{1.5}$$

holds for sufficiently small ϵ_0 in some subregions of Ω^∞ or Ω_2 for all functions ψ_3 harmonic in these regions.

Proof. Let Ω_2 be the region included between two concentric spheres of radii r_1 and r_0.

Let the center O of the spheres be the origin of coordinates and let $\phi_3 = 1/r(r^2 = x^2 + y^2 + z^2)$, a function harmonic* in Ω_2 as well as in Ω^∞.

It is clear that for this choice of ϕ_3 it is sufficient to prove the theorem for ψ_3, a function symmetric relative to the z-axis.

It is well-known that every function harmonic in Ω_2 and symmetric relative to the z-axis can be expanded in a uniformly convergent series

$$\psi_3 = \sum_{k=0}^{\infty} [a_k r^k + b_k r^{-(k+1)}]\, P_k(\cos\theta) \tag{1.6}$$

in the interior of Ω_2 (that is, for $r_1 < r < r_0$); where r, θ are spherical coordinates and $P_k(\cos\theta)$ is a Legendre polynomial. The series (1.6) can be differentiated term by term, so that

$$\frac{\partial\psi_3}{\partial z} = a_1 + \sum_{k=1}^{\infty} [(k+1)\, a_{k+1} r^k - k b_{k-1}\, r^{-(k+1)}]\, P_k(\cos\theta) \tag{1.7}$$

Furthermore, using the orthogonality of spherical functions, we obtain

$$\delta(r) = \iint_{S'} \left(\frac{\partial\psi_3}{\partial z} - \varphi_3\right) dS' = \iint_{S'} \left(\frac{\partial\psi_3}{\partial z} - \frac{1}{r}\right) dS' = 4\pi r^2 \left[a_1 - \frac{1}{r}\right] \tag{1.8}$$

where S' is the sphere of radius r and center O.

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* The function (1.4) can be used instead of this function.

It is not hard to see that

$$\min_{r_1 \le r \le r_0} \max \delta(r) > \varepsilon_0' = 2\pi r_1^2 \left(\frac{1}{r_1} - \frac{1}{r_0}\right) \tag{1.9}$$

for arbitrary choice of the coefficient a_1 of (1.7), since

$$\min_{r_1 \le r \le r_0} \max \left(a_1 - \frac{1}{r}\right) = \frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_0}\right)$$

which is assumed for

$$a_1 = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_0}\right)$$

It now follows from (1.8) and (1.9) that the integrand of (1.8) cannot be less than $\epsilon_0 = \epsilon_0'/4\pi r_0^2$ in all of Ω_2, that is,

$$\min_{r_1 < r < r_0} \max \left| \frac{\partial \psi_3}{\partial z} - \frac{1}{r} \right| > \varepsilon_0$$

in some subregion of Ω_2.

For the unbounded region Ω^∞ it is necessary to put $a_1 = 0$ in (1.5)-(1.8). Then $\delta(r) = -4\pi r \neq 0$ for all r, and (1.5) follows.

The proof of the following theorem is analogous to that of Theorem 1.

Theorem 3. In order that Neuber's solution be a general solution in a region D, it is necessary and sufficient that in an arbitrary closed region D' of D there exist harmonic functions ϕ_3^* and ψ_3^* such that

$$\frac{\partial \psi_3^*}{\partial z} = \varphi_3^*, \qquad |\varphi_3 - \varphi_3^*| < \varepsilon, \qquad \text{grad}\,|\varphi_3 - \varphi_3^*| < \varepsilon \tag{1.10}$$

where ϵ is arbitrarily small.

It now follows from Theorem 2 that Neuber's solution (1.2) is not a general solution (and hence is not complete) in either Ω^∞ or Ω_2.

Hence Neuber's solution (1.2) is not a general solution in an arbitrary unbounded domain D^∞ exterior to a closed surface S, since D^∞ always contains a subregion Ω^∞ to which Theorems 2 and 3 can be applied, with $\phi_3 = 1/r$, by taking the origin of coordinates ($r = 0$) in the interior region bounded by the surface S.

Similarly, if a doubly connected region $D^{(2)}$ contains a region Ω_2, Theorem 2 holds for $D^{(2)}$ as well, with $\phi_3 = 1/r$; consequently (1.2) will not be a general solution for such a region. However, Neuber's solution (1.2) is a general solution for an arbitrary simply connected region D_1 bounded by a Liapunov surface S_1, that is, (1.10) holds for such a region.

We shall now determine the regions D in which Neuber's solution is complete and those for which it is not, that is, we shall discuss the cases in which a solution of equations (1.3) exists and those in which it does not.

Almansi [20] proved that in order that a solution of the equations (1.3) may exist in a region T for an arbitrary harmonic function ϕ_3, it is sufficient that the region T be bounded by a surface S_T which intersects lines parallel to the z-axis in only two points. Tolotti [21] proved that this condition is also necessary.

We shall prove Tolotti's theorem (Theorem 4) for functions of the form (1.4).

Theorem 4. If a straight line parallel to the z-axis intersects the bounding surface S of a three-dimensional region D in more than two points, it is possible to construct a function ϕ_3 harmonic in D such that there is no function ψ_3 harmonic in D which satisfies (1.3).

Proof. Suppose that a straight line parallel to the z-axis intersects the surface S at points A_1, A_2, A_3, ... Choose the origin of coordinates at a point O in the exterior of the region D on the straight line $A_1A_2OA_3\ldots$, and let ϕ_3 be a function of the form (1.4).

It is clear that the function

$$\psi_3 = \sum_{m=1,2,\ldots} \left(c_m + c_m' \frac{\partial}{\partial y}\right) \frac{\partial^m}{\partial x^m} \ln(r+z) + \psi_0(x, y)$$

is harmonic in an arbitrary open bounded region V' containing no points of the negative z-axis and satisfies (1.3) ($\psi_0(x, y)$ is an arbitrary function of two variables harmonic in V'). Because of the uniqueness of the analytic continuation of a harmonic function, we infer that ψ_3 cannot be continued on all of D and has singularities at the points on the negative z-axis. Hence there is no harmonic function ψ_3 satisfying (1.3) everywhere in D, where D is a region of the indicated type and ψ_3 is an arbitrary harmonic function of the form (1.4).

It follows by Theorem 1 that Neuber's solution (1.2) is not complete, if a straight line parallel to the z-axis intersects the surface S in more than two points.

However, there is an essential difference between a doubly connected (multiply connected) and a simply connected region of the form indicated (a line parallel to the z-axis intersects the surface S in more than two points). If the region is simply connected, Neuber's solution (1.2) is a general solution; if the region is doubly (multiply) connected, (1.2) is not a general solution. In neither case is (1.2) complete.

2. Papkovich-Neuber's solution. We shall call the solution containing three harmonic functions obtained from (1.1) by eliminating ϕ_0, that is,

$$\mathbf{u} = \mathbf{B} - \nu^{-1}\,\mathrm{grad}\,(\mathbf{r}\cdot\mathbf{B}) \qquad (2.1)$$

Papkovich-Neuber's solution [2-5].

The problem of representing a displacement vector $\mathbf{u}$ satisfying the homogeneous equations of elasticity in a prescribed region D in the form (2.1), is reducible to that of determining a function ψ harmonic in D and satisfying

$$L\psi = \nu\psi - r\frac{\partial\psi}{\partial r} = \varphi_0, \qquad \nabla^2\psi = 0, \qquad \nu = 4(1-\sigma) \qquad (2.2)$$

where ϕ_0 is an arbitrary function harmonic in D. In other words, if a function ψ satisfying (2.2) in D exists, then the solution (2.1) is complete in D.

Analogously, if there is a function ψ^* harmonic in an arbitrary closed region D' of D and satisfying

$$\begin{matrix} L\psi^* = \varphi_0^*, \quad \nabla^2\psi^* = \nabla^2\varphi_0^* = 0 & \text{in} \quad D \\ \mathrm{grad}\,|\varphi_0 - \varphi_0^*| < \varepsilon & \text{in} \quad D' \end{matrix} \qquad (2.3)$$

where $\epsilon > 0$ is arbitrary, then (1.2) is a general solution in D.

It is known that there does not exist a harmonic function ψ satisfying (2.2) in all of D for an arbitrary region D and an arbitrary harmonic function ϕ_0. Moreover, the existence of ψ_0 depends on the choice of the origin of coordinates [6].

We shall prove a stronger theorem (important from the point of view of general solutions) which generalizes certain results of [6].

Theorem 5. Let Ω_2 be the region included between two concentric spheres S_1 and S_0, with S_1 the smaller sphere. Then it is possible to construct a harmonic function ϕ_0 such that

$$\int_{\Omega_2} (L[\psi] - \varphi_0)^2\, d\Omega_2 > \varepsilon_0 \qquad (2.4)$$

for sufficiently small ϵ_0 and arbitrary function ψ harmonic in Ω_2, if the origin of coordinates $(x = y = z = 0, r = 0)$ is in the exterior of the region D_1 interior to the sphere S_1.

An analogous theorem can be proved for the region Ω^∞ exterior to S_1, if the origin of coordinates is in Ω^∞.

Theorem 5 and (2.3) imply that if the origin of coordinates is chosen as indicated, then (2.1) is not a general solution and hence is not a complete solution in Ω_2.

Hence it also follows that (2.1) is not a general solution for a bounded triply connected region bounded by three disjoint spheres S_0, S_1, S_2 or for the unbounded region Ω_2^∞ exterior to two disjoint spheres S_1 and S_2, if the origin of coordinates is chosen arbitrarily.

Proof. Set $\phi_0 = \rho^{-1}$, where ρ is the distance from points of Ω_2 to the center O of the spheres* (the function ϕ_0 is harmonic in Ω_2).

Denote the radii of S_1 and S_0 by ρ_1 and ρ_0 and put $\rho_1 = 1 (\rho_1 < \rho < \rho_0)$ for simplicity.

Take the origin of coordinates $P (r = 0)$ at a distance z_0 from O on the z-axis in the direction of OP.

In addition to the coordinates x, y, z introduce coordinates x', y', z' with origin O and direct the z'-axis in the direction of OP and the x'-axis parallel to the x-axis. We have

$$x = x', \qquad y = y', \qquad z = z' - z_0$$
$$r^2 = (PM)^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + (z' - z_0)^2$$
$$\rho^2 = (OM)^2 = x'^2 + y'^2 + z'^2$$

where M is an arbitrary point of Ω_2.

Then (2.2) can be transformed into

$$L\psi = \nu\psi - \rho\frac{\partial\psi}{\partial\rho} + z_0\frac{\partial\psi}{\partial z'} = \varphi_0 \tag{2.5}$$

Because of this choice of x, y, z and x', y', z' and the symmetry of the function $\phi_0 = \rho^{-1}$ relative to the z-axis, it is clearly enough to prove (2.4) on the assumption that the function ψ of (2.4) is symmetric relative to the z- and z'-axes.

Furthermore, it is known that every function ψ harmonic in Ω_2 and symmetric with respect to the z-axis can be expanded in the interior of Ω_2 into a uniformly convergent series (1.6), with r replaced by ρ.

Consider the partial sum ψ_n (harmonic in Ω_2) of the series (1.6):

$$\psi_n = \sum_{k=0}^{n} [a_k\rho^k + b_k\rho^{-(k+1)}]\, P_k(\cos\theta) \tag{2.6}$$

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* A function of the form (1.4) can be substituted for the function $\phi_0 = \rho^{-1}$.

where ρ, θ are spherical coordinates and $P_k(\cos\theta)$ is a Legendre polynomial.

From (2.5)-(2.6) and (1.7) we obtain

$$L[\psi_n] = \sum_{k=0}^{n+1} \{[(\nu - k) a_k + z_0(k+1) a_{k+1}] \rho^k + [(\nu + k + 1) b_k - z_0 k b_{k-1}] \rho^{-(k+1)}\} P_k(\cos\theta)$$

$$a_{n+2} = a_{n+1} = b_{n+1} = b_{-1} = 0 \tag{2.7}$$

We introduce new variables a_k' and b_k'

$$(\nu - k) a_k' + z_0 (k+1) a_{k+1} = a_k' \qquad (k = 1, \ldots, n)$$
$$(\nu + k + 1) b_k - z_0 k b_{k-1} = b_k' \tag{2.8}$$

It is not hard to see that if $\nu = 4(1 - \sigma)$ is not an integer (which is true for $\sigma \neq 1/4$), the constants a_k and b_k are uniquely determined by the corresponding variables a_k' and b_k'.

From the second system of (2.8) (since $b_{-1} = b_{n+1} = 0$) we obtain

$$b_n = \frac{1}{\mu_n} \sum_{k=0}^{n} \frac{\mu_k}{\nu + k + 1} b_k', \qquad \frac{\partial b_n}{\partial b_k'} = \frac{1}{\mu_n} \frac{\mu_k}{\nu + k + 1}$$

$$\mu_k = \alpha_0 \alpha_1 \ldots \alpha_k, \qquad \mu_n = \alpha_0 \alpha_1, \ldots, \alpha_n \tag{2.9}$$

$$\alpha_0 = \nu + 1, \qquad \alpha_k = \frac{\nu + k + 1}{k} \frac{1}{z_0} \qquad (k = 1, \ldots, n)$$

Substituting (2.8) and (2.9) into (2.7) we obtain

$$L[\psi_n] - \varphi_0 = L[\psi_n] - \rho^{-1} =$$

$$= \sum_{k=0}^{n} [a_k' \rho^k + b_k' \rho^{-(k+1)}] P_k(\cos\theta) + (n+1) z_0 b_n \rho^{-(n+2)} P_{n+1}(\cos\theta) - \rho^{-1} \tag{2.10}$$

Next, substituting (2.10) into the left side of (2.4) and using the orthogonality of Legendre polynomials, we find

$$\Phi = \frac{1}{4\pi} \int_{\Omega_2} \{L[\psi_n] - \rho^{-1}\}^2 d\Omega_2 =$$

$$= \sum_{k=0}^{n} [\alpha_k a_k'^2 + 2\gamma_k a_k' b_k' + \beta_k b_k'^2] + B_n b_n^2 - 2(\gamma_0 a_0' + \beta_0 b_0') + \beta_{01} \tag{2.11}$$

Here

$$\alpha_k = [(2k+1)(2k+3)]^{-1} (\rho_0^{2k+3} - 1), \qquad \gamma_k = [2(2k+1)]^{-1} (\rho_0^2 - 1)$$
$$\beta_k = [(2k+1)(2k-1)]^{-1} (1 - \rho_0^{-2k+1}), \qquad B_n = (n+1)^2 z_0^2 \beta_{n+1} \tag{2.12}$$

We shall find the minimum value of Φ, which can be written in the form $\Phi = \Phi_2 + \Phi_1 + \Phi_0$, where Φ_2 is a quadratic form, Φ_1 is a linear form in the variables a_k', b_k' and $\Phi_0 = \beta_0$ is an independent term.

From (2.11) and (2.9) we obtain the following system of equations for determining the coefficients a_k', b_k' which yield the minimum value of Φ:

$$\frac{1}{2}\frac{\partial\Phi_2}{\partial a_k'} = \alpha_k a_k' + \gamma_k b_k' = f_k = -\frac{1}{2}\frac{\partial\Phi_1}{\partial a_k'} \tag{2.13}$$

$$\frac{1}{2}\frac{\partial\Phi_2}{\partial b_k'} = \gamma_k a_k' + \beta_k b_k' + \frac{B_n\mu_k}{(\nu+k+1)\mu_n} b_n = g_k = -\frac{1}{2}\frac{\partial\Phi_1}{\partial b_k'}$$

$$f_k = g_k = 0 \qquad (k = 1, \ldots, n)$$

$$f_0 = \gamma_0 = \frac{1}{2}(\rho_0^2 - 1), \qquad g_0 = \beta_0 = \rho_0 - 1$$

Furthermore, it is obvious that

$$\Phi = \sum_{k=0}^{n}\left[\frac{1}{2}\left(\frac{\partial\Phi_2}{\partial a_k'}a_k' + \frac{\partial\Phi_2}{\partial b_k'}b_k'\right) + \left(\frac{\partial\Phi_1}{\partial a_k'}a_k' + \frac{\partial\Phi_1}{\partial b_k'}b_k'\right)\right] + \beta_0$$

Hence, because of (2.13) we obtain

$$\Phi_{\min} = \beta_0 - \sum_{k=0}^{n}(f_k a_k' + g_k b'_k) = \beta_0 - (\gamma_0 a_0' + \beta_0 b_0') = \frac{\Delta_0}{\alpha_0}(b_0' - 1)$$

$$\Delta_0 = \begin{vmatrix} \alpha_0 & \gamma_0 \\ \gamma_0 & \beta_0 \end{vmatrix} = \alpha_0\beta_0 - \gamma_0^2, \qquad a_0' = \frac{\gamma_0}{\alpha_0} b_0' \tag{2.14}$$

By eliminating unknowns, we find from (2.13)

$$b_k' + \frac{B_n\mu_k}{\mu_n(\nu+k+1)}\frac{\alpha_k}{\Delta_k} b_n = 0 \qquad (k = 1, \ldots, n)$$

$$b_0' + \frac{B_n\mu_0}{\mu_n(\nu+1)}\frac{\alpha_0}{\Delta_0} b_n = 1 \tag{2.15}$$

$$\Delta_k = \begin{vmatrix} \alpha_k & \gamma_k \\ \gamma_k & \beta_k \end{vmatrix} = \alpha_k\beta_k - \gamma_k^2 \qquad (k = 0, 1, \ldots, n)$$

whence

$$b_k' = \frac{\nu+1}{\nu+k+1}\frac{\mu_k}{\mu_0}\frac{\alpha_k}{\alpha_0}\frac{\Delta_0}{\Delta_k}(b_0' - 1) \tag{2.16}$$

Substituting (2.16) into the second equation of (2.15) and using the expression for b_n from (2.9), we find

$$(b_0' - 1)\left[1 + \frac{B_n}{\mu_n^2}\sum_{k=0}^{n}\frac{\mu_k^2\alpha_k}{(\nu+k+1)^2\Delta_k}\right] = \frac{B_n}{\mu_n^2}\frac{\mu_0^2}{(\nu+1)^2}\frac{\alpha_0}{\Delta_0} \tag{2.17}$$

Finally, from (2.14) and (2.17) we obtain

$$\Phi_{\min} = (\nu + 1)^{-2} \left[\frac{1}{B_n} \frac{\mu_n}{\mu_0} + \sum_{k=0}^{n} \left(\frac{\mu_k}{\mu_0} \right)^2 \frac{\alpha_k}{(\nu + k + 1)^2 \Delta_k} \right]^{-1} \tag{2.18}$$

We shall investigate the expression (2.18).

We note first that $\Phi_{\min} < \beta_0 = \rho_0 - 1$, since putting $a_k' = b_k' = 0$ in (2.11) yields $b_n = 0$ by (2.9), and consequently $\Phi_{\min} < \beta_0$.

Hence the expression in square brackets in (2.18) is greater than a certain constant.

Furthermore, by (2.12)

$$\frac{1}{(\nu + k + 1)^2} \frac{\alpha_k}{\Delta_k} = \frac{(2k+1)(2k-1)}{(\nu + k + 1)^2} \left[1 - \rho_0^{-2k+1} - \frac{2k-1}{4(2k+3)} \frac{(\rho_0^2 - 1)^2}{\rho_0^{2k+3} - 1} \right]^{-1}$$

$$B_n = \frac{(n+1)^2 z_0^2}{(2n+1)(2n-1)} (1 - \rho_0^{-2k+1}) \tag{2.19}$$

so that

$$\frac{1}{(\nu + k + 1)^2} \frac{\alpha_k}{\Delta_k} \to 4, \qquad B_n \to \frac{z_0^2}{4}$$

for $\rho_\infty > 1$ and $k \to \infty$, $n \to \infty$.

From (2.9) we also find

$$\frac{\mu_k}{\mu_0} = z_0^{-k} \prod_{s=1}^{k} \frac{\nu + s + 1}{s} \tag{2.20}$$

Let $z_0 < 1$, i.e. suppose that the origin of coordinates P is in the interior of the region D_1 bounded by the sphere S_1 of radius $\rho_1 = 1$. Then it is obvious from (2.18)-(2.20) that the expression in square brackets in (2.18) can be made greater than an arbitrary positive number N by choosing a sufficiently large value of n; consequently, for sufficiently large n, $\Phi_{\min}$ will be arbitrarily small.

In other words, the series in the square brackets in (2.18) diverges for $z_0 < 1$ as $n \to \infty$ and hence $\Phi_{\min} \to 0$.

Now, suppose that $z_0 > 1$, i.e. the origin of coordinates P is in the exterior of D_1; hence P is either in Ω_2 or in the exterior of the region D_0 bounded by the sphere S_0 of radius ρ_0.

Then it follows from (2.20) that the ratio μ_n/μ_0 of (2.18) can be made less than an arbitrarily small $\epsilon > 0$ for sufficiently large n.

$$\frac{\mu_n}{\mu_0} = z_0^{-n} \prod_{s=1}^{n} \frac{\nu + s + 1}{s} < \left[\left(\frac{\nu + 1}{s_0} + 1\right) z_0^{-1}\right]^{n-s_0} \prod_{s=1}^{s_0} \frac{\nu + s + 1}{s}$$

which can be made arbitrarily small for $((\nu + 1)/s_0 + 1) z_0^{-1} < 1$ and sufficiently large n.

Furthermore, it is not hard to prove, by means of d'Alembert's criterion, that the series in square brackets in (2.18) is absolutely convergent for $n \to \infty$.

In fact, we have, by (2.20)

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\left(\frac{\mu_{n+1}}{\mu_n}\right)^2 \frac{\alpha_{n+1}\Delta_n}{\alpha_n \Delta_{n+1}} \left(\frac{\nu + n + 1}{\nu + n + 2}\right)^2 z_0^{-2}\right| =$$

$$= \lim_{n\to\infty} \left|\frac{n}{n+1} \frac{\alpha_{n+1}}{\Delta_{n+1}} \frac{\Delta_n}{\alpha_n} z_0^{-2}\right| = z_0^{-2} < 1$$

Consequently, the expression in square brackets in (2.18) is bounded from above by a number N_0 independent of n, and so

$$\Phi_{\min} > \frac{1}{(\nu + 1)^2 N_0} = \varepsilon_0 \tag{2.21}$$

which completes the proof.

Hence (2.4) holds for $z_0 > 1$. The inequality (2.4) can be proved in the same way for the region Ω^{∞} exterior to the sphere S_1, if the origin of coordinates is in Ω^{∞}. To do so, it is enough to put $a_k = 0$, $k = 0$, 1, ... in (2.6)-(2.20). It follows from Theorem 5 that if two spherical three-dimensional regions are removed from bounded or unbounded region, (2.4) holds for arbitrary choice of the origin of coordinates and therefore (2.1) is not a general solution for the resulting regions.

Furthermore, if two three-dimensional regions bounded by closed surfaces S_1' and S_2' (having no points in common with each other or with the bounding surface S_0 of D_0) and lying in the interior of the mentally excluded spherical cavities (three-dimensional regions) bounded by spherical surfaces S_1, S_2 are removed from either a bounded or unbounded region D, then (2.4) holds for the resulting regions for arbitrary choice of the origin of coordinates; consequently, (2.1) is not a general solution for these regions no matter where the origin of coordinates is chosen.

We shall now determine the domains for which Papkovich-Neuber's solution (2.1) is complete, and those for which it is not.

E. Trefftz [4] considered the system (2.2) for various coefficients ν and wrote the solution of these equations in the form

$$\psi = -r^{\nu}\int r^{-(\nu+1)}\varphi_0\,dr + Cr^{\nu} \tag{2.22}$$

where C is a constant of integration depending on the direction of the radius r. He noted that if the lower limit of integration in (2.22) is taken as $r = 0$ (the origin of coordinates lies in the interior of the region and if C is made equal to 0, then the function

$$\psi = -r^{\nu}\int_0^r r^{-(\nu+1)}\varphi_0\,dr \tag{2.23}$$

is harmonic and satisfies the equations (2.2) for $\nu < 0$ (see also Bergman [15]).

Eubanks and Sternberg [7] proved that the function (2.23) is harmonic for $\nu > 0$ (except for the values $\nu = 3$, $\sigma = 1/4$) in a region which is a star relative to the origin of coordinates; that is, a region with the property that any ray from the origin drawn in the interior of the region intersects the bounding surface of the region at precisely one point. Since the integral in (2.23) is improper for $\nu > 0$ (if the origin is in the interior of the region), Eubanks and Sternberg [7], in proving their theorem, neglected the first few terms of the series expansion of ϕ_0 about the origin in terms of spherical functions.

We shall prove first that if a ray ON, where O is the origin, intersects the bounding surface S_R of a region R in no more than two points, then the function (2.22) is harmonic for the corresponding value of the constant C. We shall also prove that if the ray ON intersects the surface $S_{R'}$ in more than two points, there cannot exist a harmonic function ψ satisfying (2.2) in the region R' for some choice of the harmonic function ϕ_0. It will then follow that (2.1) is a complete solution for R, but not for R'.

Let the origin be exterior to the region R and suppose that the ray ON intersects the surface S of R at A_1, A_2. We consider the function

$$\psi = -r^{\nu}\int_{r_1}^{r} r^{-(\nu+1)}\varphi_0\,dr + Cr^{\nu} \tag{2.24}$$

where C depends only on the spherical coordinates θ, ϕ. We write the Laplace equation in spherical coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\nabla_{\theta,\varphi}{}^2\psi = 0, \qquad \nabla_{\theta,\varphi}{}^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta}$$

where $\nabla_{\theta,\phi}{}^2$ is a differential operator depending only on the spherical coordinates θ, ϕ. Furthermore, since

$$\frac{1}{r^2}\nabla_{\theta,\varphi}^2\left[r^{\nu}\int_{r_1}^{r} r^{-(\nu+1)}\varphi_0\,dr\right] = -r^{\nu-2}\int_{r_1}^{r} r^{-(\nu+1)}\left(\frac{\partial^2\varphi_0}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi_0}{\partial r}\right)dr =$$

$$= r^{\nu-2}\left\{\left[-(\nu+1)\,r^{-\nu}\varphi_0 - r^{-\nu+1}\frac{\partial\varphi_0}{\partial r}\right]_{r_1}^{r} - \nu(\nu+1)\int_{r_1}^{r} r^{-(\nu+1)}\varphi_0\,dr\right\}$$

we obtain from (2.24):

$$\nabla^2\psi = r^{\nu-2}\{\nabla_{\theta,\varphi}^2 C + \nu(\nu+1)\,C - \theta_0\} \qquad (2.25)$$

$$\theta_0 = r_1^{-\nu}\left[(\nu+1)\,\varphi_0 + r_1\frac{\partial\varphi_0}{\partial r}\right]_{r=r_1} \qquad (2.26)$$

It is clear from (2.26) that a necessary condition for the function ψ to be harmonic in the region R is that the function $C = C(\theta, \phi)$ satisfy

$$\nabla_{\theta,\varphi}^2 C + \nu(\nu+1)\,C = \theta_0(\theta, \varphi) \qquad (2.27)$$

Vekua [19] devised a method for constructing solutions of (2.27) by means of complex functions.

The case when $\theta \to \pi/2$ requires additional investigation.

This case occurs if, for instance, R is a doubly connected finite three-dimensional region bounded by two closed surfaces S_{1R} and S_{2R}, where S_{1R} is contained in the interior of S_{2R} and the origin O is in the interior of S_{1R}, and the ray ON intersects each of the surfaces S_{1R}, S_{2R} at precisely one point.

However, the problem of constructing a function ψ, harmonic in this doubly connected domain is easily reduced to two simpler problems.

Let

$$\varphi_0 = \varphi_{01} + \varphi_{02}, \qquad \psi = \psi_1 + \psi_2$$

where ϕ_{01}, ψ_1 are functions harmonic in the region R_1^{∞} exterior to the surface S_{1R} and ϕ_{02}, ψ_2 are functions harmonic in the region R_2 interior to the surface S_{2R}. If there exist functions ψ_1, ψ_2 satisfying (2.2), with ϕ_{01}, ϕ_{02} appearing on the right-hand side of (2.2) for the regions R_1^{∞}, R_2, respectively, then there certainly exists a function ψ satisfying (2.2). The existence of the function ψ_1 follows immediately from the fact that the function (2.23), with ϕ_0 replaced by ϕ_{01} and ψ by ψ_1, satisfies equation (2.2), except for $\nu = 3$ $(\sigma = 1/4)$.

Furthermore, putting

$$\psi_2 = -r^{\nu}\int_{\infty}^{r} r^{-(\nu+1)}\varphi_{02}\,dr \tag{2.28}$$

in analogy with (2.23), it is immediately verified that ψ_2 is a harmonic function. This also follows from (2.26), since for $r_1 = \infty$, $C = 0$ and $\nabla^2\psi_2 = 0$.

Hence (2.1) is a complete solution for the region R if the origin is chosen as indicated.

We shall now prove that if the ray ON intersects the surface $S_{R'}$ of the region R' in more than two points, then there cannot exist a harmonic function ψ satisfying (2.2).

Indeed, suppose the origin O ($r = 0$) is in the interior of the region and that the ray ON intersects the surface $S_{R'}$ at points A_1, A_2, A_3, ...

Let O_1 be a point of the segment A_1A_2 in the interior of R'.

Set $\phi_0 = -1/\rho$ in (2.2), where ρ is the distance from O_1 to an arbitrary point M of the region R' ($r = OM$, $\rho = O_1M$). Since the function

$$\psi' = r^{\nu}\int_{0}^{r} r^{-(\nu+1)}\frac{1}{\rho}\,dr \tag{2.29}$$

is harmonic in the interior of the open region V' obtained by deleting the points of the half-line $O_1A_2A_3$ ... from an arbitrary bounded region V and furthermore, satisfies the equation

$$\nu\psi' - r\frac{\partial\psi'}{\partial r} = \varphi_0 = -\frac{1}{\rho} \tag{2.30}$$

in V', it is harmonic in the open region R'' obtained from R' by deleting the points of the half-line $O_1A_2A_3$ ... .

Furthermore, since the function (2.29) is (for $\sigma \neq 1/4$) the only harmonic function satisfying (2.2) in a neighborhood of the origin O, it follows from the uniqueness of the analytic continuation of a harmonic function that the function ψ' defined by (2.29) is the analytic continuation of the harmonic function satisfying (2.30) in a neighborhood of O to all of the region R''.

We shall show that ψ' is arbitrarily large in a neighborhood of the line $O_1A_2A_3$... . In fact, putting

$$OO_1 = r_0, \quad r > r_0, \quad \nu = 4(1-\sigma), \quad 0 < \sigma < \tfrac{1}{2} \qquad (\sigma \neq \tfrac{1}{4})$$

we find

$$\psi'(r,\theta) = r^{\nu}\int_0^r r^{-(\nu+1)}\frac{1}{\rho}\,dr = r^{\nu}\int_0^r r^{-(\nu+1)}\frac{dr}{\sqrt{r^2-2rr_0\cos\theta+r_0^2}} >$$

$$> r^{\nu}\,r^{-(\nu+1)}\int_{r_0\cos\theta}^{r}\frac{dr}{\sqrt{(r-r_0\cos\theta)^2+(r_0\sin\theta)^2}} =$$

$$= \frac{1}{r}\ln\left[(r-r_0\cos\theta)+\sqrt{(r-r_0\cos\theta)^2+(r_0\sin\theta)^2}\,\right]_{r_0\cos\theta}^{r} =$$

$$= \frac{1}{r}\left[\ln(r-r_0\cos\theta+\rho)-\ln(r_0\sin\theta)\right] \qquad (2.31)$$

where r, θ are spherical coordinates.

It is obvious that the right side of (2.31) is arbitrarily large as $\theta \to 0$.

It follows that there is no function ψ harmonic in all of R' and satisfying (2.30).

Hence, (2.1) is not a complete solution for R' if the origin O is in the interior of R'.

Now suppose that O is outside the region R' and that a ray ON intersects the surface $S_{R'}$ at points A_1, A_2, A_3, A_4, ... Choose a point O_1 on the segment A_2A_3 outside the region R' and as above write $\phi_0 = -1/\rho$, where $\rho = O_1M$, $r = OM$.

By adding to the right side of (2.29) an expression of the form Cr^{ν}, where C is a function depending only on the spherical coordinates θ and ϕ, we obtain the general solution of (2.30). Since the function Cr^{ν} must be harmonic in the neighborhood of the line A_1A_2, it will also be harmonic in the neighborhood of the line $O_1A_3A_4$... . Hence the function $\psi' + Cr^{\nu}$ will be arbitrarily large in the neighborhood of the line $O_1A_3A_4$... .

It follows that there is no harmonic function in R' satisfying (2.2), and so (2.1) is not a complete solution for R' either in this case or in the case when the origin is in R', as long as the ray ON intersects the surface $S_{R'}$ in more than two points.

However, there is an essential difference between the simply connected and multiply connected regions of the form R' as far as general solutions are concerned. If R' is a simply connected region bounded by a closed Liapunov surface S, (2.3) holds, i.e. (2.1) is a general solution; equation (2.3) does not hold particularly in the case of multiply connected regions (of the form R') for an arbitrary harmonic function ϕ_0 and (2.1) is not a general solution.

3. General and complete solutions for multiply connected regions. Suppose that a bounded multiply connected region D is bounded by a closed Liapunov surface S_0 and closed surfaces $S_i (i = 1, \ldots, k)$, the latter lying in the region D_0 inside S_0 and having no points in common with each other or with S_0.

It was shown in [6] that if an arbitrary harmonic vector $\mathbf{B}$ is represented in the form

$$\mathbf{B} = \mathbf{B}_0 + \sum_{i=1}^{k} \mathbf{B}_i \tag{3.1}$$

where $\mathbf{B}_0$ is a harmonic vector in D_0 and $\mathbf{B}_i$ a harmonic vector in the region D_i^{∞} exterior to the surface S_i, then

$$\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{k} \mathbf{u}_i \tag{3.2}$$

$$\mathbf{u}_i = \mathbf{B}_i - \nu^{-1}\operatorname{grad}(\mathbf{r}_i \cdot \mathbf{B}_i) \qquad (i = 1, \ldots, k)$$
$$\mathbf{u}_0 = \mathbf{B}_0 - \nu^{-1}\operatorname{grad}(\mathbf{r}_0 \cdot \mathbf{B}_0), \qquad \sigma \neq \tfrac{1}{4} \tag{3.3}$$
$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2$$

will be a general solution in D. In the formulas (3.2), (3.3), the point O_i with coordinates x_i, y_i, z_i is in the region D_i bounded by the surface S_i, and x, y, z are the coordinates of an arbitrary point of D. The vector $\mathbf{u}_0$ can also be written in Neuber's form (1.2) or in any of the other forms which are general for the region D_0 ([6], 7).

If the surface S_0 is absent, then D is an unbounded multiply connected region and we must put $\mathbf{B}_0 = 0$, $\mathbf{u}_0 = 0$ in (3.1)-(3.3).

From the proof of the generality of the solution (3.1)-(3.3) given in [6], it immediately follows that if the regions D_i are solid spheres and S_0 is a sphere, then (3.1)-(3.3) is also a complete solution for this region.

To show this we can take the limit as $n \to \infty$ in the expansions in terms of spherical polynomials of the formulas (3.1)-(3.11) and others of [6], since the series of spherical polynomials converge uniformly in the corresponding regions.

It follows further that (3.2)-(3.3) is a complete solution, if each solution (3.2) is complete for the corresponding simply connected region (the bounded region D_0 or the unbounded region D_i^{∞}). The last case occurs if, for instance, a ray O_iN intersects the surface S_i at a single point, but a ray O_0N in no more than two points (because of the restrictions in

Section 2). If the ray O_0N intersects the surface S in more than two points and the ray O_iN intersects S_i in more than one point (and hence in more than two points, since O_i is in D_i), then (3.2)-(3.3) is not complete for this region, according to Section 2.

4. Papkovich-Neuber's solution for $\sigma = 1/4$. First, it is easy to prove that if the origin is in the region and $\sigma = 1/4$, then (2.1) is not only incomplete but also not general.

Suppose, for instance, that Ω is a solid sphere in D and that the origin is the center of the sphere. Writing

$$\psi = \sum_{n=0}^{\infty} \rho^n Y_n, \qquad \varphi_0 = \rho^3 Y_3', \quad \sigma = \frac{1}{4}, \ \nu = 4(1-\sigma) = 3 \tag{4.1}$$

in (2.2) and (2.4), where Y_n is the spherical function of order n, we obtain

$$\int_\Omega [L(\psi) - \varphi_0]^2 d\Omega = \int_\Omega \Big[\sum_{n=0}^{\infty} (3-n)\rho^n Y_n - \rho^3 Y_3'\Big]^2 d\Omega =$$

$$= \sum_{n=0}^{\infty} \int_\Omega [(3-n)\rho^n Y_n]^2 d\Omega + \varepsilon_0 \geqslant \varepsilon_0, \qquad \varepsilon_0 = \int_\Omega [\rho^3 Y_3']^2 d\Omega \tag{4.2}$$

It follows from (4.2) that (2.1) is not a general solution for $\sigma = 1/4$ if the origin is in D. Strangely enough, however, if the origin is outside the sphere Ω, then (2.1) is a general and complete solution for Ω and for arbitrary $0 < \sigma < 1/2$. This follows immediately from (2.24)-(2.29).

We consider the interesting special case, $\phi_0 = r^3 Y_3' = r^3 P_3(\cos\theta)$, where r is the distance from P to a point M of Ω and the positive z-axis is directed by PO (O is the center of the sphere). In this case, a solution of

$$\nu\psi - r\frac{\partial\psi}{\partial r} = \varphi_0 = r^\nu P_\nu, \qquad \nu = 3 \tag{2.2}$$

where P_ν is the generalized Legendre function, is immediately found without having recourse to (2.24)-(2.29). The solution, due to Bromwich [22], is

$$\psi = \frac{\partial}{\partial\nu}(r^\nu P_\nu)$$

For $\nu = 3$ this function has the form

$$\psi = r^3 \ln\frac{r+z}{2} P_3(\cos\theta) - 2r^3\left\{\frac{5}{6}P_2(\cos\theta) - \frac{3}{10}P_1(\cos\theta) + \frac{1}{12}\right\} \tag{4.3}$$

It is easily and immediately verified that (4.3) satisfies (2.2) for

$\nu = 3$ and is harmonic in Ω (that is, for $z > 0$).

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